Hilbert's Theorem 90 for partial actions

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Let L/k be a finite Galois extension of fields with Galois group $G = Gal(L/k)$. The original statement of Hilbert's Theorem 90 says that if $G = \langle \sigma \rangle$, then an element $x \in L$ has norm

$$N(x) = \prod_{i=1}^{|G|} \sigma^i(x) = 1,$$

if and only if,

$$x = \frac{\sigma(y)}{y}, \quad \text{for some } y \in L.$$

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A generalization in terms of cohomology was found by Emmy Noether. In cohomological language, Hilbert's Theorem 90 states that

$$H^1(Gal(L/k),\mathcal{U}(L)) = 0,$$

for any finite Galois extension of fields L/k.

An extension of this result to the context of Galois extension of commutative rings was obtained in

AG The Brauer group of a commutative ring. *Trans. Amer. Math. Soc.* (1960) 52: 367-409.

Theorem

Let S be a Galois extension of R relative to G. If every finitely generated projective R-module of rank one is free, then $H^1(G,\mathcal{U}(S)) = (0)$.

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The purpose of this talk is to extend the Theorem above to the context of *Partial Galois extensions*.

Contents

Partial Galois theory

DFP M. Dokuchaev, M. Ferrero, A. Paques, Partial actions and Galois theory, *J. Pure Appl. Algebra* (2007) 208: 77-87.

Let $\alpha = (D_g, \alpha_g)_{g\in G}$ be a unital partial action of a finite group G on a commutative ring R, and write

$$D_g = R1_g, \quad 1_g^2 = 1_g,\ g \in G.$$

The subring of *invariants* of R under α is

$$R^\alpha = \{r \in R \,|\, \alpha_g(r1_{g^{-1}}) = r1_g, \text{ for all}\, g \in G\}.$$

The ring extension R/R^α is called an *α-partial Galois extension*, if for some $m \in \mathbb{N}$ there exists a subset $\{x_i, y_i \,|\, 1 \le i \le m\}$ of R such that

$$\sum_{i=1}^{m} x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g}, \quad g \in G.$$

The set $\{x_i, y_i \,|\, 1 \le i \le m\}$ is called an *α-partial Galois coordinate system.*

Given a unital partial action $\alpha = (D_g, \alpha_g)_{g\in G}$ one may define the *partial skew group ring* , which is the abelian group

$$R \star_\alpha G = \bigoplus_{g\in G} D_g\delta_g,$$

where the δ_g' s are symbols.

The multiplication in $R \star_\alpha G$ is induced by

$$(r_g\delta_g)(t_h\delta_h) = r_g\alpha_g(t_h 1_{g^{-1}})\delta_{gh}.$$

Then there is a ring monomorphism

$$R \ni r \overset{\iota}{\mapsto} r\delta_1 \in R \star_\alpha G,$$

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For any $R \star_\alpha G$-module M we denote

$$M^G = \{m \in M \mid (1_g\delta_g)m = 1_g m, \text{ for all } g \in G\}.$$

Then

Theorem (DFP)

The following statements are equivalent.

- *R/R^α is a partial Galois extension;*
- *R is a f.g projective R^α-module and the map $\mu\colon R \otimes_{R^\alpha} M^G \to M$, given by*

$$\mu(x \otimes_{R^\alpha} m) = xm, \text{ for all } x \in R, m \in M$$

  is an isomorphism of R-modules.

Partial cohomology of groups

We recall form

DK M. Dokuchaev, M. Khrypchenko, Partial cohomology of groups, arXiv:1309.7069.

Definition

Let R be a commutative ring, $n \in \mathbb{N}, n \geq 1$ and $\alpha = (D_g, \alpha_g)_{g \in G}$ an unital partial action of G on R. A n-cochain of G with values in R is a function $f : G^n \to R$, such that

$$f(g_1, \ldots, g_n) \in \mathcal{U}(R1_{g_1}1_{g_1g_2} \ldots 1_{g_1g_2\ldots g_n}),$$

A 0-cochain is an invertible element of R.

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Denote

$$C^n(G,\alpha,R) = \{f\colon G^n \to R \mid f \text{ is an } n\text{-cochain}\}.$$

It is an abelian group under point-wise multiplication. Its identity is

$$G^n \ni (g_1, g_2, \ldots, g_n) \mapsto 1_{g_1} 1_{g_1 g_2} \cdots 1_{g_1 \ldots g_n} \in \mathcal{U}(R1_{g_1} 1_{g_1 g_2} \ldots 1_{g_1 g_2 \ldots g_n}).$$

Definition (The coboundary operator)

For any $f \in C^n(G,\alpha,R)$ *and* $g_1,\dots,g_{n+1} \in G$ *set*
$(\delta^n f)(g_1,\dots,g_{n+1}) =$

$$\alpha_{g_1}\left(f(g_2,\dots,g_{n+1})1_{g_1^{-1}}\right)\prod_{i=1}^{n} f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})^{(-1)^i} f(g_1,\dots,g_n)^{(-1)^{n+1}}.$$

Here the inverse elements are taken in the corresponding ideals.
If $n=0$ *and* $r \in \mathcal{U}(R) = C^0(G,\alpha,R)$, *we set*

$$(\delta^0 r)(g) = \alpha_g(1_{g^{-1}}r)r^{-1}, \quad \text{for all } g \in G.$$

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Proposition (DK)

$\delta^n : C^n(G,\alpha,R) \to C^{n+1}(G,\alpha,R)$, is a homomorphism such that

$$(\delta^{n+1}\delta^n f)(g_1, g_2, \dots, g_{n+2}) = 1_{g_1} 1_{g_1 g_2} \dots 1_{g_1 g_2 \dots g_{n+2}},$$

for any $f \in C^n(G,\alpha,R)$.

It follows that for every $n \in \mathbb{N}, n \geq 1$ one has

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Definition

We define the groups

- $Z^n(G,\alpha,R) = \ker\delta^n$, *of partial n-cocycles.*
- $B^n(G,\alpha,R) = \operatorname{im}\delta^{n-1}$, *partial n-coboundaries.*
- $H^n(G,\alpha,R) = \dfrac{\ker\delta^n}{\operatorname{im}\delta^{n-1}}$, *partial n-cohomologies.*

For $n = 0$ *we define* $H^0(G,\alpha,R) = Z^0(G,\alpha,R) = \ker\delta^0$.

The first partial cohomology group

For example, $H^1(G, \alpha, R) = \dfrac{Z^1(G, \alpha, R)}{B^1(G, \alpha, R)}$

and

$$H^1(G, \alpha, R) = \frac{\{f \in C^1 \mid f(gh)1_g = f(g)\,\alpha_g(f(h)1_{g^{-1}}), \forall g, h \in G\}}{\{f \in C^1 \mid f(g) = \alpha_g(r1_{g^{-1}})r^{-1},\ \text{for some } r \in \mathcal{U}(R)\}}.$$

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The Picard group of a commutative ring

Let S be a commutative ring and P be a f.g.p S-module. It is known that for any $\mathfrak{p} \in \mathrm{Spec} S$, the localization

$$P_{\mathfrak{p}} = P \otimes_S S_{\mathfrak{p}}$$

is a f.g free $S_{\mathfrak{p}}$-module.

Then

$$P_{\mathfrak{p}} \cong S_{\mathfrak{p}}^{n_{\mathfrak{p}}} \quad \text{as} \quad S_{\mathfrak{p}}\text{-modules}, \quad \text{for some } n_p \in \mathbb{N}.$$

Thus we get a function $\varphi_P \colon \mathrm{Spec} S \ni \mathfrak{p} \mapsto n_{\mathfrak{p}} \in \mathbb{N}$.

When $\varphi_P = 1$, that is $P_{\mathfrak{p}} \cong S_{\mathfrak{p}}$, for all $\mathfrak{p} \in \mathrm{Spec} S$, we say that P has *rank* 1, and write $\mathrm{rk}_S(P) = 1$.

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Now let

$$\mathbf{Pic}(S) = \{[P] \mid \mathrm{rk}_S(P) = 1\}.,$$

where

$$[P] = \{M \in {}_S\mathcal{M}od \mid M \cong P \text{ as } S\text{-modules}\},$$

is the isomorphism class of a module P.

The set $\mathbf{Pic}(S)$ is an abelian group, by the operation

$$[P][Q] = [P \otimes_S Q].$$

- The identity in $\mathbf{Pic}(S)$ is $[S]$.
- $[P]^{-1} = [P^*]$, where $P^* = \mathrm{Hom}_S(P, S)$.

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Hilbert's Theorem 90 for partial actions

Theorem

Let α be a unital partial action of a finite group G on a commutative ring R. Suppose that R/R^α is a partial Galois extension and $\mathbf{Pic}(R^\alpha) = (0)$, then $H^1(G, \alpha, R) = 0$.

Remark

Some rings with trivial Picard group are:

- *semi-local rings,*
  - *local rings,*
  - *fields,*
  - *finite rings.*

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The Theorem above is a consequence of the following.

Proposition

Let α be a unital partial action of a finite group G on a commutative ring R. Suppose that R/R^{α} is a partial Galois extension. Then there is a group monomorphism $H^1(G,\alpha,R) \to \mathrm{Pic}(R^{\alpha})$.

Sketch of the Proof. Let $f \in Z^1(G,\alpha,R)$, we define a $R\star_{\alpha} G$-module R_f by

- $R_f = R$ as sets,
- $(r_g\delta_g)\cdot r = r_g f(g)\alpha_g(r1_{g^{-1}})$, for any $r \in R$, $g \in G$.

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Sketch of the Proof. (continuation) Since R/R^α is a partial Galois extension, there is a R-module isomorphism

$$R \otimes_{R^\alpha} R_f^G \cong R_f = R. \tag{1}$$

Using that R is a f.g projective R^α-module and (1) one concludes that R_f^G is a f.g projective R^α-module and $\mathrm{rk}_{R^\alpha}(R_f^G) = 1$.

From this one gets a map

$$\varphi\colon H^1(G,\alpha,R) \ni \mathrm{cls}(f) \to [R_f^G] \in \mathrm{Pic}(R^\alpha),$$

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Hilbert's 90 for partial actions

If $\mathrm{Pic}(R^{\alpha}) = 0$, we have an exact sequence

$$0 \longrightarrow H^1(G, \alpha, R) \longrightarrow 0.$$

Which implies

$$H^1(G, \alpha, R) = 0,$$

and we have our version of Hilbert's 90.

Thanks